Scattering of guided waves in a waveguide with a slightly rough boundary: Stochastic functional approach

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The guided normal waves or modes in a waveguide would be perturbed if the boundaries of the waveguide were to become statistically irregular or rough. Due to the accumulation effects of multiple scattering along the entire propagation path, even very slight boundary irregularities can give rise to a considerable influence on the propagation characteristics of the guided modes. In this paper, a way is proposed to treat the scattering problem of guided waves in a waveguide with a slightly rough boundary by applying the stochastic functional approach, which has been used successfully in the scattering problems of a plane scalar or electromagnetic wave in free space from various shaped random rough surfaces and has been shown to be good for treating the multiple scattering effects. As a prototype of the basic theory, only the planar structure of the waveguide and the Dirichlet boundary condition are considered. The waveguide's Green's function is expanded in terms of the Wiener-Hermite stochastic functionals of a homogeneous Gaussian random surface. Expressions for the modified normal waves (modes) of the average or coherent Green's function are given for the Dirichlet boundary condition. A mass operator is derived which contains the information of the multiple scattering of the modes from the rough boundary and can be evaluated in an iterative way. The second order statistical moment or the correlation function of the Green's function is also considered. Some numerical examples are given for illustration. It has been shown that our approach gives more thorough results than those given by the graphical or Feynman diagram method.

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I. INTRODUCTION

The scattering of waves from a random rough surface is a problem not only of theoretical interest but also of practical importance, and at the same time a very common physical phenomenon [1–3]. In view of its physical phenomena, the problem can evidently be divided into two groups [2]. The first group is related to interactions of the waves in free spaces or half-spaces with rough surfaces; for instance, the scattering of radio waves from irregular ground or sea surfaces, the wave diffraction from a rough body, or the excitation of surface plasmons in random metal surfaces. Common to all these situations is that the wave field interacts with only a finite portion of the surface, namely a single act of scattering from the rough surface. Afterwards, the scattered waves travel in free space and never again interact with the boundary irregularities. For this reason, only slight distortions of the wave field can be produced if the perturbation of the boundary irregularity is small enough, and, as a result, even the first Born approximation of perturbation theory gives satisfactory solutions.

The second group is related to interactions of guided waves in waveguides or standing waves in resonant cavities with the rough boundaries inside. The natural examples of such waveguides are the earth-ionosphere cavity, the underwater acoustic channel, tropospheric ducts, and most importantly, the telecommunication waveguides (for instance, optical fibers), where more or less random deviations from the ideal (cylindrical) cross section can occur. The obvious results from the effects of small boundary roughness are to increase the attenuation and to decrease the coherence of the modes [4]. Since the energy of the guided waves is mainly bounded in the domain that contains the rough boundaries, the guided waves will undergo again and again scattering from the irregularities as they are propagating along the boundaries. Hence, the wave field at an observation point is the sum of the waves multiply scattered by the irregularities distributed along the entire propagation path. Even very slight boundary perturbations can give rise to considerable distortions in the field pattern due to accumulation effects, which, though they may be harmless in short-distance propagation, have to be given more attention in long-distance (international) communication systems.

Considering such effects in the framework of conventional perturbation theory, which respects only a single act of scattering, is evidently not very fruitful, and has been shown merely to have a narrow range of validity. It is clear then that, for the effective treatment of the second group of phenomena, the theories that are used have to

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allow summation of the waves that are repeatedly rescattered by small irregularities of the boundaries. Bass, Fuks, and co-worker [2,5,6] have studied in detail this kind of problem by the graphical or Feynman diagrammatic method in a way similar to the one in random media, and given some important and significant results such as the average Green's function and its second order statistical moment as well as the radiation transfer equations for the modes' intensities. It can be seen that the equations they used for solving the average Green's function and the second order moment bear, respectively, a very close resemblance to the Dyson equation and the Bethe-Salpeter (BS) equation in the propagation theory in random media [7]. Tolstoy [8,9] has investigated the coherent modes and boundary waves in a rough-walled acoustic waveguide, in a really different way. In his theory, many small scatterers are used to model the roughness elements of a rough surface, which is usually called the boss model that was first proposed by Twersky [10]. The most important result obtained by Tolstoy is that the so-called boundary waves exist under certain conditions in his boss model. DeSanto [11] has also treated the problem of an ocean waveguide with the randomly rough upper boundary and obtained the equivalent impedance for the rough boundary, based on the Green's function expression in which a so-called phase modulation angular spectral term is introduced to describe the rough surface interaction. It seems that his result can be applicable to both small and large roughness, but on the other hand, it is not in an explicit form. It should be noted that the common point that all the methods mentioned above share is that the rough surface or boundary can be made equivalent by a linear boundary condition applied to a smoothed surface, namely the smoothed boundary conditions, which have been widely used for the first group of problems [12,13].

In the present paper, we intend to deal with the problem of propagation and the scattering of guided waves in a waveguide with a slightly rough boundary by virtue of the stochastic functional approach. The approach was first introduced in the theory of propagation in random media by the first author [14–16], and has been used successfully to develop the scattering theory of a plane scalar or electromagnetic wave from various planar [17–22], cylindrical [23–25], and spherical [26] random rough surfaces with small roughness. In these works, the scattered wave field is regarded as a stochastic functional of the random surface that can be represented in the form of a Wiener-Hermite expansion [27,28] in the case of a Gaussian random surface, and a group-theoretic consideration is made to determine the form of a stochastic wave field based on the statistical homogeneity of the random surface, which is analogous to the Floquet theorem for a periodic boundary. A set of hierarchical equations for the expansion coefficients is obtained from the boundary conditions and can be solved by making use of the recurrence relations and the orthogonality of the Wiener-Hermite functionals. Various statistical characteristics of the scattered waves, such as coherent and incoherent fields, their differential cross section (the second order moments) and angular distribution, etc., can be easily calculated. More importantly, it has been shown that the so-called divergence difficulty in the common perturbation theory, which is due to the multiple scattering in the direction close to the planar random surface, is automatically removed in our approach owing to the "stochastic Floquet theorem" and the stochastic functional calculus [19]. This means that the stochastic functional approach is good enough for treating the multiple scattering effects, and hence can be applied to the scattering problems of the guided waves. In fact, the influence of the guided waves on the scattering properties of a rough surface has been considered in a previous paper [20] in which the excitation of surface plasmons (modes) in an Ag film with rough surface was studied for the incident plane wave from outside.

We begin this investigation by examining a planar waveguide with a bottom rough boundary, and consider merely the Dirichlet boundary condition for the Green's function. There is no mathematical or physical difficulty in extending the theory developed in this paper to cases of more complex geometry and other kinds of boundary conditions. In this sense, the present paper is a prototype of the basic theory. We have ordered the paper as follows. In Sec. II, a stochastic representation of the Green's function in the planar waveguide is presented in terms of the complex Wiener-Hermite functionals of a homogeneous Gaussian random surface. Then, in Sec. III, we determine the Wiener expansion coefficients according to the Dirichlet boundary condition, and obtain the modified normal waves of the average or coherent Green's function by the residue evaluation. Furthermore, the second order statistical moment or the correlation function of the Green's function is discussed in Sec. IV, especially touching on the intensities of the guided modes. Finally, some numerical examples and brief conclusions are given in Sec. V.

II. STOCHASTIC REPRESENTATION OF GREEN'S FUNCTION

Let us consider a waveguide with planar structure, as shown in Fig. 1. For an easy comparison with others' works [2,5,6], we also deal with the Green's function of the waveguide. If both the boundaries are smooth or unperturbed, then the unperturbed Green's function $G_0(\mathbf{R},\mathbf{R}')$ can be expressed as [5,29]

$$G_0(\mathbf{R},\mathbf{R}')=G_0(\mathbf{r},\mathbf{r}',z,z')=\frac{1}{(2\pi)^2}\int d\boldsymbol{\lambda}_0 \exp[i\boldsymbol{\lambda}_0\cdot(\mathbf{r}-\mathbf{r}')]\tilde{G}_0(\boldsymbol{\lambda}_0,z,z')\ , \tag{1}$$

where $\mathbf{r}=(x,y)$ and $\mathbf{r}'=(x',y')$ are the two-dimensional radius vectors of the field point $\mathbf{R}=(x,y,z)$ and the source point $\mathbf{R}'=(x',y',z')$, respectively. $\boldsymbol{\lambda}_0$ and its amplitude or absolute value $\lambda_0=|\boldsymbol{\lambda}_0|$ denote the wave vector and the propagation constant. $\tilde{G}_0(\boldsymbol{\lambda}_0,z,z')$ in Eq. (1) can be regarded as the two-dimensional Fourier transform of the Green's function $G_0(\mathbf{R},\mathbf{R}')$ and satisfies the inhomogeneous wave equation

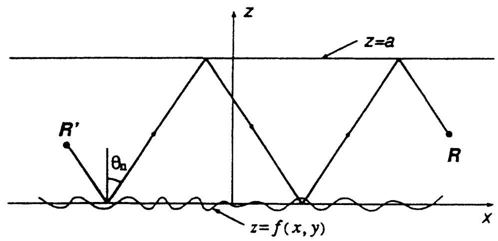

FIG. 1. A planar waveguide with the bottom rough boundary.

$$\left[\frac{d^2}{dz^2}+k_0^2\epsilon(z)-\lambda_0^2\right]\tilde{G}_0(\boldsymbol{\lambda}_0,z,z')=-\delta(z-z') , \quad (2)$$

where k_0 is the wave number in free space and $\epsilon(z)$ is an appropriate profile function standing for the material property filling the inside of the waveguide. According to the Sturm-Liouville theory for the ordinary differential equation with certain boundary conditions, the solution of Eq. (2) can be constructed as

$$\tilde{G}_0(\boldsymbol{\lambda}_0,z,z')=\Phi_1(\lambda_0,z_<)\Phi_2(\lambda_0,z_>)/W(\lambda_0,z') , \quad (3)$$

where $z_<$ and $z_>$ denote the lesser and greater of z or z', respectively, and in the case under discussion $0\le z, z'\le a$. The functions $\Phi_1(\lambda_0,z_<)$ and $\Phi_2(\lambda_0,z_>)$ are the linearly independent solutions of the homogeneous equation

$$\left[\frac{d^2}{dz^2}+k_0^2\epsilon(z)-\lambda_0^2\right]\Phi(\lambda_0,z)=0 \quad (4)$$

and satisfy the boundary conditions at the bottom and top boundaries, respectively. In the case of the Dirichlet condition, they are $\Phi_1(\lambda_0,0)=0$ and $\Phi_2(\lambda_0,a)=0$. $W(\lambda_0,z)=\Phi_1'(\lambda_0,z)\Phi_2(\lambda_0,z)-\Phi_1(\lambda_0,z)\Phi_2'(\lambda_0,z)$ is the Wronskian determinant, in which the primes denote the first derivative with respect to z: $\Phi'(\lambda,z)=d\Phi(\lambda,z)/dz$. It has been well known [29] that $W(\lambda_0,z)$ is not dependent on z and is really the function of λ_0 only, so if we set $z=0$ and have $\Phi_1(\lambda_0,0)=0$ in mind, we then reduce to $W(\lambda_0)=W(\lambda_0,z)=W(\lambda_0,0)=\Phi_1'(\lambda_0,0)\Phi_2(\lambda_0,0)=\Phi_1'(\lambda_0)\Phi_2(\lambda_0)$ [we will use $\Phi(\lambda_0)$ to denote $\Phi(\lambda_0,0)$ hereafter], which is in the same form as that used in [2,5]. It should be noted that the nulls of $W(\lambda_0)$ stand for the propagation constants of the normal waves or modes in the waveguide. Hence, by using the residue theorem to evaluate the integration of Eq. (1), we can obtain the expression of $G_0(\mathbf{R},\mathbf{R}')$ in terms of the summation of the normal waves.

We now convert to consider the rough-walled waveguide, in which for simplicity only the bottom boundary is assumed as a random rough surface expressed by $z=f(x,y)$, where $f(x,y)$ is a random function with the mean $\langle f(x,y)\rangle=0$. If $f(x,y)$ is a homogeneous Gaussian random function, then as shown in previous papers [17–26] we have the spectral representation of $f(x,y)$ by a Wiener integral as

$$z=f(x,y)=f(\mathbf{r})=\int \exp[i\boldsymbol{\lambda}\cdot\mathbf{r}]F(\boldsymbol{\lambda})dB(\boldsymbol{\lambda}) , \quad (5)$$

where $dB(\boldsymbol{\lambda})$ denotes the complex Gaussian random measure in two-dimensional space, namely, a complex random variable with the properties

$$\langle dB(\boldsymbol{\lambda})\rangle=0,\quad dB^*(\boldsymbol{\lambda})=dB(-\boldsymbol{\lambda}) ,$$
$$\langle dB(\boldsymbol{\lambda})dB^*(\boldsymbol{\lambda}')\rangle=\delta(\boldsymbol{\lambda}-\boldsymbol{\lambda}')d\boldsymbol{\lambda}d\boldsymbol{\lambda}' , \quad (6)$$

where the angle brackets $\langle\ \rangle$ denote the probabilistic average over the sample space of the random functions or variables, and the asterisk, the complex conjugate. From Eq. (5) and by making use of Eq. (6), we have the following expressions for the correlation function:

$$R(\mathbf{r})=\langle f(\mathbf{r}+\mathbf{r}_0)f(\mathbf{r}_0)\rangle$$
$$=\int \exp[i\boldsymbol{\lambda}\cdot\mathbf{r}]|F(\boldsymbol{\lambda})|^2d\boldsymbol{\lambda} \quad (7)$$

and the variance that describes the random surface roughness

$$\sigma^2=R(0)=\int |F(\boldsymbol{\lambda})|^2d\boldsymbol{\lambda} , \quad (8)$$

where we have used the relation $F(\boldsymbol{\lambda})=F^*(-\boldsymbol{\lambda})$. $|F(\boldsymbol{\lambda})|^2$ is called the power spectrum of the random surface. $|F(\boldsymbol{\lambda})|^2=0$ and then $\sigma^2=0$ corresponds to a smooth or flat boundary.

It is obvious that the Green's function will be perturbed and become random, as the bottom boundary is statistically a rough surface. If the perturbation is slight, the perturbed Green's function can then be written as

$$G(\mathbf{R},\mathbf{R}')=G_0(\mathbf{R},\mathbf{R}')+G_s(\mathbf{R},\mathbf{R}') , \quad (9)$$

where $G_s(\mathbf{R},\mathbf{R}')$ is introduced to represent the effect of the rough boundary and can be expressed as a stochastic functional of the random surface function $f(x,y)$. Furthermore, if we suppose that $f(x,y)$ is a homogeneous Gaussian random function, then just as has been done in previous papers [17–26], we can expand, in view of the fact that the random wave field is the eigenfunction of a shift operator $D^{\mathbf{r}}$ (defined by a translation in the x-y plane and a measure-preserving transformation in the sample space; see [24] for details) with the eigenvalue $\exp[i\boldsymbol{\lambda}\cdot\mathbf{r}]$ and the form of $G_0(\mathbf{R},\mathbf{R}')$ in Eq. (1), the scattered Green's function $G_s(\mathbf{R},\mathbf{R}')$ in terms of the Wiener-Hermite functionals in the following manner:

$$G_s(\mathbf{R},\mathbf{R}')=\frac{1}{(2\pi)^2}\int d\boldsymbol{\lambda}_0\exp[i\boldsymbol{\lambda}_0\cdot(\mathbf{r}-\mathbf{r}')]\Phi_2(\lambda_0,z)\Phi_2(\lambda_0,z')A_0(\boldsymbol{\lambda}_0)$$
$$+\frac{1}{(2\pi)^2}\sum_{n=1}^{\infty}\int\cdots\int d\boldsymbol{\lambda}_0\exp[i(\boldsymbol{\lambda}_0+\boldsymbol{\lambda}_1+\cdots+\boldsymbol{\lambda}_n)\cdot\mathbf{r}-i\boldsymbol{\lambda}_0\cdot\mathbf{r}']$$
$$\times\Phi_2(\eta_n,z)\Phi_2(\lambda_0,z')A_n(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_n)\hat{h}_n(dB(\boldsymbol{\lambda}_1),dB(\boldsymbol{\lambda}_2),\ldots,dB(\boldsymbol{\lambda}_n)) , \quad (10)$$

where $\hat{h}_n(\ )$ denotes the nth order complex Wiener-Hermite differential, which is to be understood as a generalization of the Hermite polynomial (note that $\hat{h}_0=1$), the integral in Eq. (10) represents the n-tuple complex Wiener integral, and the coefficients A_n are the unknown integral kernels to be determined by applying the boundary condition on the rough boundary. The parameter $\eta_n=|\boldsymbol{\lambda}_0+\boldsymbol{\lambda}_1+\cdots+\boldsymbol{\lambda}_n|$ is the length of a composed wave vector $\boldsymbol{\eta}_n=\boldsymbol{\lambda}_0+\boldsymbol{\lambda}_1+\cdots+\boldsymbol{\lambda}_n$, which originates from the scattering from the rough boundary. The reason for choosing $\Phi_2(z)$ rather than $\Phi_1(z)$ is to satisfy the boundary condition at the flat boundary $z=a$, and the reason for choosing $\Phi_2(z')$ is to make $G_s(\mathbf{R},\mathbf{R}')$ satisfy a homogeneous wave equation (source free). Equation (9) together with Eq. (10) can be regarded as a stochastic representation of the Green's function in the waveguide under discussion, which is the stochastic functional of a homogeneous Gaussian random surface.

On the other hand, by averaging Eq. (10), we can also express the Green's function as the sum of two parts, namely, the coherent and incoherent part. The coherent or average Green's function (the first-order statistical moment) is given by

$$G_c(\mathbf{R},\mathbf{R}')=\langle G(\mathbf{R},\mathbf{R}')\rangle=G_0(\mathbf{R},\mathbf{R}')+G_{s0}(\mathbf{R},\mathbf{R}')\ , \quad (11)$$

where $G_{s0}(\mathbf{R},\mathbf{R}')$ is the first term in Eq. (10) and represents the contribution from the coherent scattering. The incoherent Green's function $G_{\text{inc}}(\mathbf{R},\mathbf{R}')$ is then obtained by subtracting $G_c(\mathbf{R},\mathbf{R}')$ from $G(\mathbf{R},\mathbf{R}')$, that is, the second term (the sum part) in the right hand side of Eq. (10).

III. MODIFIED NORMAL WAVES

To investigate the Green's function from Eqs. (9) and (10) in detail, we have to determine the Wiener expansion coefficients A_n by applying the boundary condition at the random boundary $z=f(x,y)$. For simplicity and only to demonstrate the usefulness of the stochastic functional approach, we confine ourselves here to dealing with the case in which the random boundary is slightly rough, that is, $\sigma^2\ll 1$; then the boundary condition for the Dirichlet problem can be approximated as

$$G+f\frac{\partial G}{\partial z}\bigg|_{z=0}=0\ . \quad (12)$$

Substituting the expressions of G_0 and G_s into Eq. (12), and making use of the recurrence formula and the orthogonality relation for $\hat{h}_n$ [17–26,30],

$$\begin{aligned}dB(\boldsymbol{\lambda})\hat{h}_n(dB(\boldsymbol{\lambda}_1),dB(\boldsymbol{\lambda}_2),\ldots,dB(\boldsymbol{\lambda}_n))\\ =\hat{h}_{n+1}(dB(\boldsymbol{\lambda}),dB(\boldsymbol{\lambda}_1),dB(\boldsymbol{\lambda}_2),\ldots,dB(\boldsymbol{\lambda}_n))\\ +\sum_{i=1}^{n}\hat{h}_{n-1}(dB(\boldsymbol{\lambda}_1),\ldots,dB(\boldsymbol{\lambda}_{i-1}),dB(\boldsymbol{\lambda}_{i+1}),\ldots,dB(\boldsymbol{\lambda}_n))\delta(\boldsymbol{\lambda}+\boldsymbol{\lambda}_i)d\boldsymbol{\lambda}d\boldsymbol{\lambda}_i\ ,\end{aligned} \quad (13)$$

$$\langle\hat{h}_n(dB(\boldsymbol{\lambda}_{i_1}),\ldots,dB(\boldsymbol{\lambda}_{i_n}))\hat{h}_m(dB(\boldsymbol{\lambda}_{j_1}),\ldots,dB(\boldsymbol{\lambda}_{j_m}))\rangle=\delta_{nm}\delta_{ij}^{n}d\boldsymbol{\lambda}_{i_1}\cdots d\boldsymbol{\lambda}_{i_n}d\boldsymbol{\lambda}_{j_1}\cdots d\boldsymbol{\lambda}_{j_m}\ , \quad (14)$$

where δ_{nm} is Kronecker's delta and δ_{ij}^n denotes the sum of all distinct products of n delta functions of the form $\delta(\boldsymbol{\lambda}_{i_\nu}+\boldsymbol{\lambda}_{j\mu})$, $i=(i_1,i_2,\ldots,i_n)$, $j=(j_1,j_2,\ldots,j_m)$, all i_ν and j_μ appearing just once in each product, we consequently obtain a set of hierarchical equations for the Wiener coefficients as follows:

$$m=0:\ \Phi_2(\lambda_0)A_0(\boldsymbol{\lambda}_0)+\int\Phi_2'(\eta_1)A_1(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1)F^*(\boldsymbol{\lambda}_1)d\boldsymbol{\lambda}_1=0\ , \quad (15)$$

$$m=1:\ \Phi_2(\eta_1)A_1(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1)+[\Phi_1'(\lambda_0)/W(\lambda_0)+\Phi_2'(\lambda_0)A_0(\boldsymbol{\lambda}_0)]F(\boldsymbol{\lambda}_1)+2\int\Phi_2'(\eta_2)A_2(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)F^*(\boldsymbol{\lambda}_2)d\boldsymbol{\lambda}_2=0\ , \quad (16)$$

$$\begin{aligned}m=2:\ \Phi_2(\eta_2)A_2(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)+[\Phi_2'(\eta_1)A_1(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1)F(\boldsymbol{\lambda}_2)+\{\Phi_2'(\eta_1^{(1)})A_1^{(1)}(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_2)F(\boldsymbol{\lambda}_1)\}]/2\\ +3\int\Phi_2'(\eta_3)A_3(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3)F^*(\boldsymbol{\lambda}_3)d\boldsymbol{\lambda}_3=0\ ,\end{aligned} \quad (17)$$

$\vdots$

$$\begin{aligned}m=n-1:\ \Phi_2(\eta_{n-1})A_{n-1}+[\Phi_2'(\eta_{n-2})A_{n-2}F(\boldsymbol{\lambda}_{n-1})]/(n-1)+\sum_{i=1}^{n-2}\{[\Phi_2'(\eta_{n-2}^{(i)})A_{n-2}^{(i)}F(\boldsymbol{\lambda}_i)]/(n-1)\}\\ +n\int\Phi_2'(\eta_n)A_n(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_n)F^*(\boldsymbol{\lambda}_n)d\boldsymbol{\lambda}_n=0\ ,\end{aligned} \quad (18)$$

$$\begin{aligned}m=n:\ \Phi_2(\eta_n)A_n+[\Phi_2'(\eta_{n-1})A_{n-1}F(\boldsymbol{\lambda}_n)]/n+\sum_{i=1}^{n-1}\{[\Phi_2'(\eta_{n-1}^{(i)})A_{n-1}^{(i)}F(\boldsymbol{\lambda}_i)]/n\}\\ +(n+1)\int\Phi_2'(\eta_{n+1})A_{n+1}(\boldsymbol{\lambda}_0,\ldots,\boldsymbol{\lambda}_{n+1})F^*(\boldsymbol{\lambda}_{n+1})d\boldsymbol{\lambda}_{n+1}=0\ ,\end{aligned} \quad (19)$$

where $\eta_m^{(i)}$ and $A_m^{(i)}$ have the same expressions as η_m and A_m, except for $\boldsymbol{\lambda}_i$, which is replaced by $\boldsymbol{\lambda}_{m+1}$. By neglecting A_{n+1} in Eq. (19), A_n is obtained in terms of $A_{n-1}(A_{n-1}^{(i)})$. Furthermore, by inserting the so-obtained A_n into Eq. (18), in principle, $A_{n-1}(A_{n-1}^{(i)})$ can be given in terms of $A_{n-2}(A_{n-2}^{(i)})$. Unfortunately, we find that in this case Eq. (18) becomes an integral equation because of the existence of $A_{n-1}^{(i)}$, and it is difficult to get the explicit solutions for the A_m's. On the other hand, however, if we discard at first the $A_m^{(i)}$'s (which merely represent some of the higher-order interactions) in each order of equations (the parts inside { }), we can obtain a set of iteratively solvable equations for A_m's in an explicit form. The final approximate solutions are

$$A_0=\frac{M(\eta_0)}{\Phi_2(\lambda_0)[\Phi_2(\lambda_0)-\Phi_2'(\lambda_0)M(\eta_0)]}\,, \tag{20}$$

$$A_1=-\frac{[\Phi_1'(\lambda_0)/W(\lambda_0)+\Phi_2'(\lambda_0)A_0(\lambda_0)]}{[\Phi_2(\eta_1)-\Phi_2'(\eta_1)M(\eta_1)]}F(\lambda_1)=\frac{-F(\lambda_1)}{[\Phi_2(\lambda_0)-\Phi_2'(\lambda_0)M(\eta_0)][\Phi_2(\eta_1)-\Phi_2'(\eta_1)M(\eta_1)]}\,, \tag{21}$$

$$A_n=-\frac{\left[\Phi_2'(\eta_{n-1})A_{n-1}F(\lambda_n)+\sum_{i=1}^{n-1}\Phi_2'(\eta_{n-1}^{(i)})A_{n-1}^{(i)}F(\lambda_i)\right]\Big/ n}{[\Phi_2(\eta_n)-\Phi_2'(\eta_n)M(\eta_n)]}\quad (n\geq 2)\,, \tag{22}$$

where the mass operator $M(\eta_n)$ satisfies the iterative equation

$$M(\eta_n)=\int\frac{\Phi_2'(\eta_{n+1})|F(\lambda_{n+1})|^2 d\lambda_{n+1}}{[\Phi_2(\eta_{n+1})-\Phi_2'(\eta_{n+1})M(\eta_{n+1})]}\,, \tag{23}$$

in which $M(\eta_{n+1})$ in the denominator of the integral kernel reflects the contributions from higher orders.

By substituting A_0 of Eq. (20) into Eq. (11), it is easy to show that the coherent or average Green's function has the following expression (notice that $\eta_0=\lambda_0$):

$$G_c(\mathbf{R},\mathbf{R}')=\frac{1}{(2\pi)^2}\int d\lambda_0\exp[i\lambda_0\cdot(\mathbf{r}-\mathbf{r}')]\times\frac{\Psi_1(\lambda_0,z_<)\Phi_2(\lambda_0,z_>)}{\Psi_1'(\lambda_0)[\Phi_2(\lambda_0)-\Phi_2'(\lambda_0)M(\lambda_0)]}\,, \tag{24}$$

with

$$\Psi_1(\lambda_0,z)=[\Phi_2(\lambda_0)-\Phi_2'(\lambda_0)M(\lambda_0)]\Phi_1(\lambda_0,z)+\Phi_1'(\lambda_0)M(\lambda_0)\Phi_2(\lambda_0,z) \tag{25}$$

satisfying the "impedance" boundary condition at $z=0$

$$\Psi_1(\lambda_0)=\Psi_1'(\lambda_0)M(\lambda_0)=W(\lambda_0)M(\lambda_0)=\Phi_1'(\lambda_0)\Phi_2(\lambda_0)M(\lambda_0)\,. \tag{26}$$

In the derivation of Eq. (24), we have used the relation $W(\lambda_0)=\Phi_1'(\lambda_0)\Phi_2(\lambda_0)$ for the Dirichlet boundary. It can be noted that Eq. (24) is exactly in the same form as Eq. (2.11) given by Bass, Freulicher, and Fuks [5]. As pointed out in [5], by analogy with the "smooth" waveguide, it seems natural to get the normal waves or modes of the coherent field by evaluating the residues on the roots of the denominator factor in Eq. (24):

$$\Phi_2(\lambda_0)-\Phi_2'(\lambda_0)M(\lambda_0)=0\,, \tag{27}$$

which indicates that the dispersion equation of the original smooth or flat waveguide $\Phi_2(\lambda_0)=0$ has been modified due to the influence of the bottom rough boundary. It should be pointed out that, as can be seen in Eq. (26), it seems that $\Psi_1'(\lambda_0)$ involves the unperturbed dispersion factor $\Phi_2(\lambda_0)$. However, with $\Phi_1(\lambda_0)\equiv 0$ in mind, it is convenient to rewrite

$$\Phi_1(\lambda_0,z)=\Phi_2(\lambda_0)\psi_1(\lambda_0,z)-\psi_1(\lambda_0)\Phi_2(\lambda_0,z)\,, \tag{28}$$

where $\psi_1(\lambda_0,z)$ $[\psi_1(\lambda_0)=\psi_1(\lambda_0,0)]$ is also a solution of Eq. (4) and independent of $\Phi_2(\lambda_0,z)$. Inserting Eq. (28) into Eq. (25), we then obtain

$$\Psi_1(\lambda_0,z)=\Phi_2(\lambda_0)\{[\Phi_2(\lambda_0)\psi_1(\lambda_0,z)-\psi_1(\lambda_0)\Phi_2(\lambda_0,z)]-M(\eta_0,z)[\Phi_2'(\lambda_0)\psi_1(\lambda_0,z)-\psi_1'(\lambda_0)\Phi_2(\lambda_0,z)]\}\,, \tag{29}$$

which means that, in Eq. (24), $\Phi_2(\lambda_0)$ in the denominator can be cancelled. Thus we need merely to evaluate the residue contributions from the nulls of Eq. (27) to get the normal waves in Eq. (24), just as was done by Bass, Freulicher, and Fuks [5]. For the isotropic rough surface, we have

$$F(\lambda)=F(\lambda),\quad M(\lambda)=M(\lambda)\,; \tag{30}$$

then the expression of the modified normal waves from Eq. (24) is

$$G_c(\mathbf{R},\mathbf{R}')=\frac{i}{2}\sum_n\frac{\tilde\beta_n H_0^{(1)}(\tilde\beta_n|\mathbf{r}-\mathbf{r}'|)\Psi_1(\tilde\beta_n,z_<)\Phi_2(\tilde\beta_n,z_>)}{\Psi_1'(\tilde\beta_n)\left.\frac{d}{d\lambda_0}[\Phi_2(\lambda_0)-\Phi_2'(\lambda_0)M(\lambda_0)]\right|_{\lambda_0=\tilde\beta_n}}\,, \tag{31}$$

with $\tilde{\beta}_n$ standing for the roots of the dispersion equation (27), namely the perturbed propagation constants of the respective modified normal waves. If the perturbation is small enough, the perturbed propagation constants can be regarded as small corrections to the unperturbed ones:

$$\delta\beta_n=\tilde{\beta}_n-\beta_n=\Phi_2'(\beta_n)\left[\frac{M(\lambda_0)}{d\Phi_2(\lambda_0)/d\lambda_0}\right]\Bigg|_{\lambda_0=\beta_n}, \tag{32}$$

where β_n is the root of $\Phi_2(\lambda_0)=0$ and stands for the unperturbed propagation constants.

It is not difficult to demonstrate that, if we neglect $M(\eta_1)$ in the denominator of Eq. (23) for $n=0$, our mass operator $M(\lambda_0)$ then corresponds to the one given in [5] under the Bourret approximation (the first-order approximation). However, it is clear that in our approach it is easy to include the contributions from the higher-order terms, by using Eq. (23) to evaluate $M(\lambda_0)$ in an iterative way, and obtain better values. Otherwise, although the mass operator used by Bass and co-workers [2,5] is also represented by an infinite series, in their method it is difficult to get the analytical expressions for the higher-order terms, or even for the second-order term.

IV. SECOND-ORDER STATISTICAL MOMENT

The second-order statistical moments of a random wave field or the Green's function are obviously related to power characteristics such as intensity, energy flux, etc. Bass and co-workers [2,6] obtained the Bethe-Salpeter (BS) type equation for the correlation function of the Green's function by introducing the so-called intensity operator. However, the BS equation would have been a poor instrument for treating the problem of scattering from the irregular surfaces because of its mathematical complexity. It is very difficult to get a general solution of the BS equation, even with the formally known intensity operator. Moreover, with the simplest form of the intensity operator, the BS equation can be solved only in some extreme cases and for specific correlation functions of the perturbations. In contrast to their method, one can see that with our approach, it is very convenient to obtain the second- or even higher-order field's statistical moments of the Green's function because of the orthogonal properties of the Wiener-Hermite differentials.

The correlation function or the second-order statistical moment of the Green's function is defined by

$$\begin{aligned}\mathcal{J}(\mathbf{R},\mathbf{R}',\mathbf{R}_0,\mathbf{R}_0')&=\langle G(\mathbf{R},\mathbf{R}')G^*(\mathbf{R}_0,\mathbf{R}_0')\rangle\\&=\mathcal{J}_c+\mathcal{J}_{\text{inc}}\\&=\langle G_c(\mathbf{R},\mathbf{R}')\rangle\langle G_c^*(\mathbf{R}_0,\mathbf{R}_0')\rangle\\&\quad+\langle G_{\text{inc}}(\mathbf{R},\mathbf{R}')G_{\text{inc}}^*(\mathbf{R}_0,\mathbf{R}_0')\rangle\ .\end{aligned} \tag{33}$$

According to the results given in the preceding sections, it is easy to obtain

$$\mathcal{J}_c=\frac{1}{(2\pi)^4}\int\int d\boldsymbol{\lambda}_0 d\boldsymbol{\lambda}_0'\exp[i\boldsymbol{\lambda}_0\cdot(\mathbf{r}-\mathbf{r}')-i\boldsymbol{\lambda}_0'\cdot(\mathbf{r}_0-\mathbf{r}_0')]\times\left[\frac{\Psi_1(\lambda_0,z_<)\Phi_2(\lambda_0,z_>)}{\Psi_1'(\lambda_0)[\Phi_2(\lambda_0)-\Phi_2'(\lambda_0)M(\lambda_0)]}\right]\left[\frac{\Psi_1(\lambda_0',z_{0<})\Phi_2(\lambda_0',z_{0>})}{\Psi_1'(\lambda_0')[\Phi_0(\lambda_0')-\Phi_2'(\lambda_0')M(\lambda_0')]}\right], \tag{34}$$

$$\begin{aligned}\mathcal{J}_{\text{inc}}&=\frac{1}{(2\pi)^4}\sum_{p=1}^{\infty}\int\int d\boldsymbol{\lambda}_0 d\boldsymbol{\lambda}_0'\int\int d\boldsymbol{\lambda}_1 d\boldsymbol{\lambda}_1'\cdots\int\int d\boldsymbol{\lambda}_p d\boldsymbol{\lambda}_p'\exp[i(\boldsymbol{\eta}_p\cdot\mathbf{r}-\boldsymbol{\eta}_p'\cdot\mathbf{r}')-i(\boldsymbol{\lambda}_0\cdot\mathbf{r}_0-\boldsymbol{\lambda}_0'\cdot\mathbf{r}_0')]\\&\quad\times\langle\hat{h}_p(dB(\boldsymbol{\lambda}_1),\ldots,dB(\boldsymbol{\lambda}_p))\hat{h}_p^*(dB(\boldsymbol{\lambda}_1'),\ldots,dB(\boldsymbol{\lambda}_p'))\rangle A_p(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_p)A_p^*(\boldsymbol{\lambda}_0',\boldsymbol{\lambda}_1',\ldots,\boldsymbol{\lambda}_p')\\&\quad\times\Phi_2(\eta_p,z)\Phi_2(\lambda_0,z')\Phi_2(\eta_p',z_0)\Phi_2(\lambda_0',z_0')\end{aligned} \tag{35}$$

$$\begin{aligned}&=\frac{1}{(2\pi)^4}\sum_{p=1}^{\infty}\int\int d\boldsymbol{\lambda}_0 d\boldsymbol{\lambda}_0'\exp[i\boldsymbol{\lambda}_0\cdot(\mathbf{r}-\mathbf{r}')-i\boldsymbol{\lambda}_0'\cdot(\mathbf{r}_0-\mathbf{r}_0')]\\&\quad\times\int d\boldsymbol{\lambda}_1\cdots d\boldsymbol{\lambda}_p\exp[i(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2+\cdots+\boldsymbol{\lambda}_p)\cdot(\mathbf{r}-\mathbf{r}_0)]\\&\quad\times(p!)|A_p(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_p)|^2\Phi_2(\eta_p,z)\Phi_2(\lambda_0,z')\Phi_2(\eta_p',z_0)\Phi_2(\lambda_0',z_0'),\end{aligned} \tag{36}$$

where $\boldsymbol{\eta}_p=\boldsymbol{\lambda}_1+\cdots+\boldsymbol{\lambda}_p$, $\boldsymbol{\eta}_p'=\boldsymbol{\lambda}_0'+\boldsymbol{\lambda}_1'+\cdots+\boldsymbol{\lambda}_p'$ in Eq. (35) and $\boldsymbol{\eta}_p'=\boldsymbol{\lambda}_0'+\boldsymbol{\lambda}_1+\cdots+\boldsymbol{\lambda}_p$ in Eq. (36). For the derivation of Eq. (36) from Eq. (35), we have used the symmetry of the Wiener kernal $A_p(\boldsymbol{\lambda}_0,\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_p)$ [see Eq. (22)] with respect to its arguments $(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\ldots,\boldsymbol{\lambda}_p)$ (invariant under exchange of the arguments). Based on such symmetry, δ_{ij}^p in Eq. (14) can be regarded as $\delta_{ij}^p\sim(p!)\delta(\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_1')\delta(\boldsymbol{\lambda}_2-\boldsymbol{\lambda}_2')\cdots\delta(\boldsymbol{\lambda}_p-\boldsymbol{\lambda}_p')$.

It is too complicated to give the explicit or meaningful expressions for a general waveguide, so for simplicity we will restrict ourselves to the case in which the waveguide is homogeneous in the y direction (independent of y, so it is only a one-dimensional propagation problem) and with $\epsilon(z)=$const, and in this case $\Phi_2(\eta,z)=\sin[\sqrt{k^2-\eta^2}(z-a)]$ (where $k^2=k_0^2\epsilon$). Furthermore, although we can deal with the case of $\mathbf{R}_0\neq\mathbf{R}$ and $\mathbf{R}_0'\neq\mathbf{R}'$, we will merely consider the case of $\mathbf{R}_0=\mathbf{R}$ and $\mathbf{R}_0'=\mathbf{R}'$ because the value $I(\mathbf{R},\mathbf{R}')=\mathcal{J}(\mathbf{R},\mathbf{R}',\mathbf{R},\mathbf{R}')$ evidently means the intensity at $\mathbf{R}$ with a source at $\mathbf{R}'$. Thus, from Eqs. (34) and (35), we can obtain

$$I_c=\langle G_c(\mathbf{R},\mathbf{R}')\rangle\langle G_c^*(\mathbf{R},\mathbf{R}')\rangle$$
$$=\frac{1}{(2\pi)^2}\int\int d\lambda_0 d\lambda_0' \exp[i(\lambda_0-\lambda_0')(x-x')]$$
$$\times\left[\frac{\sin(h_0 z_<)\sin[h_0(z_>-a)]}{h_0[\sin(h_0a)+h_0\cos(h_0a)M(\lambda_0)]}\frac{\sin(h_0' z_<)\sin[h_0'(z_>-a)]}{h_0'[\sin(h_0'a)+h_0'\cos(h_0'a)M(\lambda_0')]}\right], \quad (37)$$

$$I_{\rm inc}=\langle G_{\rm inc}(\mathbf{R},\mathbf{R}')G_{\rm inc}^*(\mathbf{R},\mathbf{R}')\rangle$$
$$=\frac{1}{(2\pi)^2}\sum_{p=1}^{\infty}\int\int d\lambda_0 d\lambda_0'\int\int d\lambda_1 d\lambda_1'\cdots\int\int d\lambda_p d\lambda_p'$$
$$\times\exp[i(\eta_p-\eta_p')x-i(\lambda_0-\lambda_0')x']\prod_{j=1}^{p}[F(\lambda_j)F^*(\lambda_j')\delta(\lambda_j-\lambda_j')]$$
$$\times\left[\frac{\sin[h_p(z-a)]\sin[h_0(z'-a)]\prod_{j=1}^{p-1}h_j\cos(h_ja)}{\prod_{j=0}^{p}\{\sin(h_ja)+h_j\cos(h_ja)M(\eta_j)\}}\right]\left[\frac{\sin[h_p'(z-a)]\sin[h_0'(z'-a)]\prod_{j=1}^{p-1}h_j'\cos(h_j'a)}{\prod_{j=0}^{p}\{\sin(h_j'a)+h_j'\cos(h_j'a)M(\eta_j')\}}\right], \quad (38)$$

where $h_j=\sqrt{k^2-\eta_j^2}$, $h_j'=\sqrt{k^2-(\eta_j')^2}$, and $\eta_j=\lambda_0+\lambda_1+\cdots+\lambda_j$, $\eta_j'=\lambda_0'+\lambda_1'+\cdots+\lambda_j'$. It should be noted that we have written Eq. (38) in the form of Eq. (35) rather than Eq. (36) for deriving approximation results similar to those of Bass and co-workers [2,6], as shown below. It is not easy to give simpler expressions from Eq. (36), particularly for the higher-order terms and in the case where there are many propagation modes in the waveguide, but we believe that Eq. (36) is better for carrying out the numerical calculations to get more rigorous results, owing to the reduced orders of integration in it. By making use of the following relations:

$$\delta(\lambda-\lambda')=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp[i(\lambda-\lambda')x]dx \quad (39)$$

and

$$\int d\eta\frac{f(\eta)\exp[\pm i\eta x]}{\sin(\sqrt{k^2-\eta^2}a)+\sqrt{k^2-\eta^2}\cos(\sqrt{k^2-\eta^2}a)M(\eta)}\simeq(i2\pi)\sum_{n=-N}^{N}\frac{q_n}{|\beta_n a|}f(\beta_n)\exp[\pm i\beta_n x-\gamma_n x]\Theta(nx) \quad (40)$$

to evaluate the residue contributions from the denominator factors in Eqs. (37) and (38), we finally obtain

$$I_c\simeq\sum_{n=-N}^{N}\sum_{l=-N}^{N}\frac{\sin q_n z\,\sin q_l z\,\sin q_n z'\,\sin q_l z'}{a^2|\beta_n\beta_l|}g_{nl}(x-x'), \quad (41)$$

$$I_{\rm inc}\simeq\sum_{n=-N}^{N}\sum_{l=-N}^{N}\frac{\sin q_n z\,\sin q_l z}{a^2|\beta_n\beta_l|^{1/2}}\sum_{i=1}^{\infty}I_{\rm inc}^{(i)}, \quad (42)$$

$$I_{\rm inc}^{(i)}=\sum_{\mu_1=-N}^{N}\sum_{\nu_1=-N}^{N}\cdots\sum_{\mu_i=-N}^{N}\sum_{\nu_i=N}^{N}\frac{\sin q_{\mu_i}z'\,\sin q_{\nu_i}z'}{|\beta_{\mu_i}\beta_{\nu_i}|^{1/2}}w_{n\mu_1}^{l\nu_1}w_{\mu_1\mu_2}^{\nu_1\nu_2}\cdots w_{\mu_{i-1}\mu_i}^{\nu_{i-1}\nu_i}$$
$$\times\int dx_1\int dx_2\cdots\int dx_i g_{nl}(x-x_1)g_{\mu_1\nu_1}(x_1-x_2)\cdots g_{\mu_i\nu_i}(x_i-x'), \quad (43)$$

and

$$w_{\mu\mu'}^{\nu\nu'}=\left[\frac{2\pi}{a^2}\right]\frac{q_\mu q_{\mu'}q_\nu q_{\nu'}}{|\beta_\mu\beta_{\mu'}\beta_\nu\beta_{\nu'}|^{1/2}}F(\beta_\mu-\beta_{\mu'})F^*(\beta_\nu-\beta_{\nu'}), \quad (44)$$

$$g_{\mu\nu}(x)=\exp[i(\beta_\mu-\beta_\nu)x-(\gamma_\mu+\gamma_\nu)x]\Theta(\mu x)\Theta(\nu x), \quad (45)$$

where $\Theta(x)$ is the step function [$\Theta(x)=1$ if $x>0$, otherwise $\Theta(x)=0$], $q_n=(n\pi/a)=k\cos\theta_n$ (we have also used $\sin q_n a = 0$ in the derivation), and $\beta_n = \mathrm{sgn}(n)\sqrt{k^2 - q_n^2} = \mathrm{sgn}(n) k \sin\theta_n$ (see Fig. 1 for the meaning of θ_n) are, respectively, the transverse eigenvalue (wave number) and the longitudinal propagation constant of the nth mode (with the positive and negative values of n denoting the forward and backward modes, respectively) in the unperturbed waveguide; N denotes the number of propagational modes $[N\pi/a < k < (N+1)\pi/a]$. The factor γ_n is the imaginary part of $\delta\beta_n$ [that is, $\gamma_n = \mathrm{Im}\delta\beta_n$, see Eq. (32) for the definition of $\delta\beta_n$] and means physically the wave damping arising from the incoherent transformation of every normal mode into other modes (including both the forward and backward directions). From Eq. (32), we get the expression of γ_n for the case under discussion as

$$\gamma_n = \sum_{m=-N}^{N} \left[\frac{\pi}{a^2}\right] \frac{q_n^2 q_m^2}{|\beta_n \beta_m|} F(\beta_n - \beta_m)|^2$$

$$= \frac{1}{2} \sum_{m=-N}^{N} w_{nm}^{nm} . \tag{46}$$

It can be found that Eqs. (41)–(43) merely give the field's intensity rather than the modal intensity. To obtain the summation of intensity of each mode, Bass *et al.* [2,6] have used an average procedure by integrating $I(\mathbf{R},\mathbf{R}')$ over an interval $\mathcal{L}$ (the representative scale length of the irregularity-caused processes over which the mode conversions occur significantly) to exclude its "rapid" oscillations due to interference between the modes (shown by the factor $\exp[i(\beta_n - \beta_l)x]$ in Eqs. (41) and (43) when $n \neq l$). For reaching such averaging, here we only need to allow in Eqs. (41)–(43)

$$g_{nl}(x) = \bar{g}_{nl}(x)\delta_{nl} = \exp(-2\gamma_n x)\Theta(nx)\delta_{nl} . \tag{47}$$

Comparing the so-obtained I_c and I_{inc} with those given in [6], we find there are more terms in our results. These terms are the ones in Eq. (43) with $\mu_j \neq \nu_j$, $j = 1, 2, \ldots, i$. They reflect the contributions to the intensity of the nth mode coming from the interactions between the different modes excited by the same point source at $\mathbf{R}'$ and seem to correspond to a portion of the cross-connected parts of the Feynman diagram in the graphical method [see Eq. (37.2) in [2]]. Nevertheless, because of the oscillating property of the factor, $\exp[(\beta_{\mu_j} - \beta_{\nu_j})x]$, with $\mu_j \neq \nu_j$ in the integral kernels of Eq. (43), these terms are much smaller than the terms with $\mu_j = \nu_j$, so we can usually neglect them for obtaining simpler results. Thus, Eqs. (41)–(43) will be reduced to the analogous form of Eqs. (1.7) and (1.8) in [6]; then we can similarly get the intensity transfer equation as

$$\mathrm{sgn}(n)\frac{d\bar{I}_n}{dx} = -2\gamma_n \bar{I}_n + \sum_{m=-N}^{N} w_{nm}^{nm} \bar{I}_m \tag{48}$$

$$= \sum_{m=-N}^{N} w_{nm}^{nm}(\bar{I}_m - \bar{I}_n) , \tag{49}$$

where $\bar{I}_n$ denotes the intensity of the nth propagation mode. Equation (49) is directly obtained by inserting Eq. (46) into Eq. (48). It should be pointed out that Eq. (1.11) in [6] seems to have no meaning according to Eq. (2.22) in [5]. In fact, from Eq. (46) it is seen that the so-called optical theorem (the energy conservation law—no "real" energy dissipation occurs in the system but just the energy conversions between the modes) is automatically satisfied under the approximation considered now.

V. NUMERICAL EXAMPLES AND CONCLUSIONS

For a numerical calculation we conveniently assume that the power spectrum of the random boundary has the Gaussian form

$$|F(\lambda)|^2 = (\sigma^2 l/\sqrt{\pi})\exp(-\lambda^2 l^2) , \tag{50}$$

where l denotes the correlation length. The spectrum is a decreasing function of λ and has a maximum at $l = 1/\sqrt{2}\lambda$ as a function of l. These properties determine the certain effects of the rough boundary on the propagating characteristics of the modes.

We can calculate γ_n and w_{nm} from Eqs. (46) and (50). We have normalized the results by a factor a^3/σ^2 because they are directly proportional to the roughness σ^2 as long as it satisfies the condition $\sigma^2 \ll a^2$. For convenience, a normalized coupling coefficient $\alpha_{nm} = (a^3/\sigma^2)(w_{nm}^{nm}/2)$ is defined for the nth and mth modes.

Figure 2 shows the normalized damping coefficient of the lowest-order mode $(a^3/\sigma^2)\gamma_1$ as a function of the normalized waveguide size ka/π for the different values of the correlation length kl. At the integer values of ka/π, γ_1 has an infinite damping in accordance with the approximation Eq. (32), so that the approximation seems to be invalid in these cases. For obtaining more accurate results, Eq. (27) has to be solved directly. In addition, from Eq. (49) it can be seen that the self-coupling coefficient $\alpha_{11}(\alpha_{nn})$ has no effect on the coupling between the mode intensities. It just brings out the loss of coherence rather than the real damping of the average or

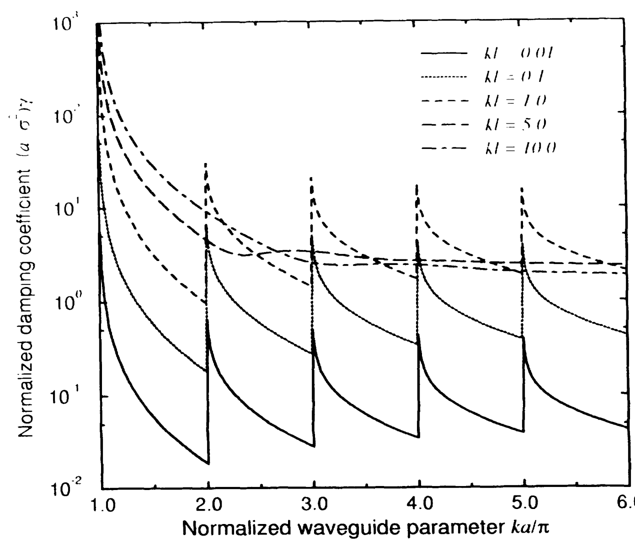

FIG. 2. Normalized damping coefficient of the lowest-order mode $(a^3/\sigma^2)\gamma_1$ versus normalized waveguide size ka/π for $kl = 0.01$, 0.1, 1.0, 5.0, and 10.0.

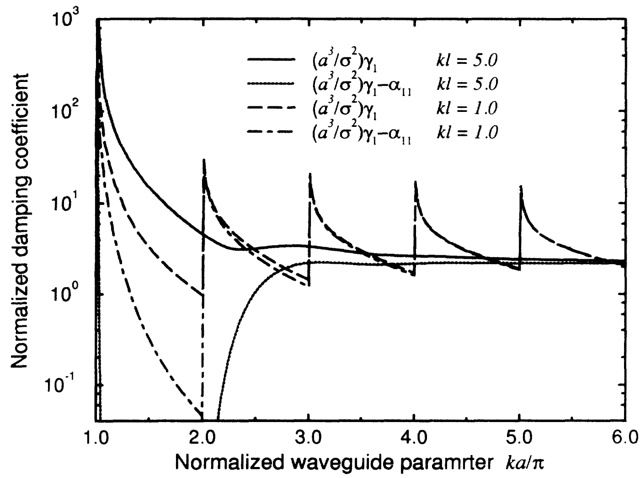

FIG. 3. Normalized damping coefficient of the lowest-order mode $(a^3/\sigma^2)\gamma_1$ and $(a^3/\sigma^2)\gamma_1-\alpha_{11}$ versus normalized waveguide size ka/π for $kl=1.0$ and 5.0.

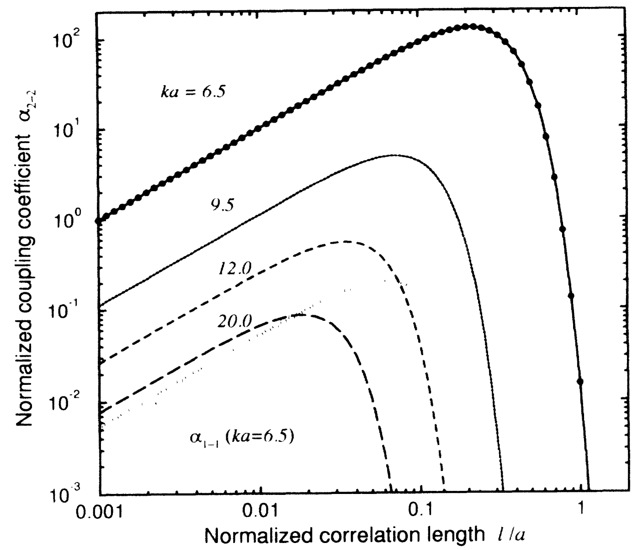

FIG. 5. Normalized coupling coefficient of the second-order forward-backward modes α_{2-2} versus normalized correlation length l/a for $ka=6.5$, 9.5, 12.0, and 20.0.

coherent field. Physically we may understand that it stands for the energy exchange caused by the nonpropagating modes while a wave is scattered into the "specular" direction from the rough boundary. In Fig. 3 we compare $(a^3/\sigma^2)\gamma_1$ with $(a^3/\sigma^2)\gamma_1-\alpha_{11}$ as a function of ka/π for different values of kl. Their differences decrease as ka increases and kl decreases. As the modes tend to cutoff (corresponding to where ka/π tends to an integer), the differences become larger, because in these cases the number of the multiple scattering of unit length from the rough boundary also increases rapidly ($\theta_n \rightarrow 0$ in Fig. 1).

Figures 4–7 show the various α_{nm} as a function of the normalized correlation length l/a for the different values of ka. We can see that the coupling coefficient of the forward-forward modes α_{nm} is always larger than that of the forward-backward modes α_{n-m}, because the difference between their propagation constants in the former is always smaller than that in the latter (the backward mode has a negative propagation constant). As ka increases, the coupling between two of any modes always decreases. For a fixed ka, the coupling between the higher-order modes is larger than that between the lower-order modes. It is also seen that there is a value of l/a giving out the largest coupling. This is due to the form of the spectrum, Eq. (50). From Eqs. (46) and (50), the point at which the coupling coefficient α_{nm} arrives at a maximum is $l/a=0.707/(\beta_n-\beta_m)a$.

When $1<ka<2$, only the first-order modes (both the forward and backward direction) can propagate in the

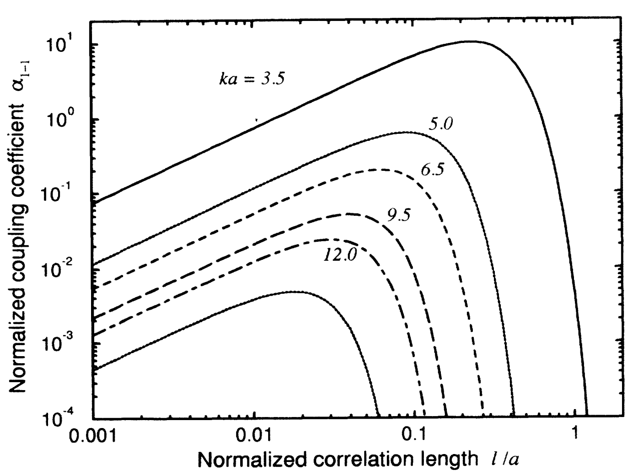

FIG. 4. Normalized coupling coefficient of the lowest-order forward-backward modes α_{1-1} versus normalized correlation length l/a for $ka=3.5$, 5.0, 6.5, 9.5, 12.0, and 20.0.

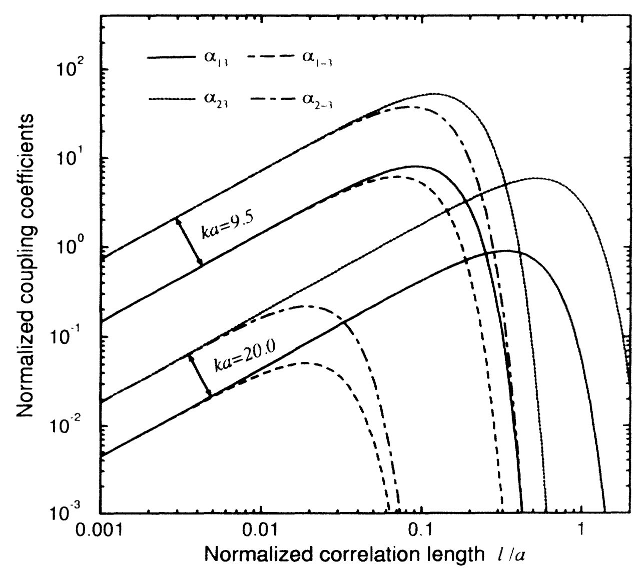

FIG. 6. Normalized coupling coefficients of the first-order forward to the second-order forward modes α_{12} and the first-order forward to the second-order backward modes α_{1-2} versus normalized correlation length l/a for $ka=6.5$, 9.5, and 20.0.

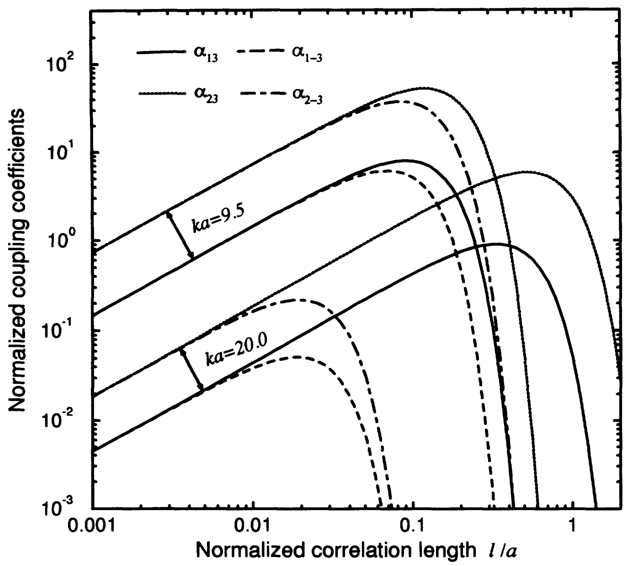

FIG. 7. Same as in Fig. 6 but for the first-order to the third-order modes α_{13} and α_{1-3} and the second-order to the third-order modes α_{23} and α_{2-3} for $ka=9.5$ and 20.0.

waveguide (corresponding to the case $N=1$). To get a feeling for the magnitude of the loss of a forward mode's intensity, expected due to the coupling or transformation of the energy from the forward to backward mode caused by the rough boundary in the case, we consider the worst possible value $\alpha_{1-1}=0.629$ at $l/a=0.091$ for $ka=5.0$ (see Fig. 4). By neglecting the recoupling or retransformation of the backward to forward mode [that is, Eq. (49) reduces to $dI_1/dx=w_{1-1}^{1-1}I_1$], we can roughly estimate the values of the intensity losses of a forward-propagating mode directly from the results of w_{1-1}^{1-1}. With the numerical results given above, we obtain that the losses are 68.7 dB/km at the wavelength $\lambda_0=1$ cm ($k=2\pi/\lambda_0$, so $a=0.8$ cm for $ka=5.0$) and 6.87 dB/km at $\lambda_0=10$ cm ($a=8$ cm), respectively, as the normalized rms roughness $\sigma/a=0.01=1.0\%$. Conversely, a normalized rms roughness σ/a of 0.21% at $\lambda_0=1$ cm and 0.66% at $\lambda_0=10$ cm will cause a loss of 3 dB/km. These values show that even a very slight boundary roughness can give rise to considerable influences on the propagation characteristics of the guided modes.

In conclusion, we have proposed a way to treat the scattering problem of the guided waves in a waveguide with a statistically slight rough boundary, by applying the stochastic functional approach. Some numerical examples are also given for illustration. It has been shown that our approach gives more thorough results than those given by the graphical or diagrammatic (Feynman diagrams) method [2,5,6]. Although the random rough boundary is assumed to obey a homogeneous Gaussian distribution statistically, it is not difficult to extend the theory to other distributions; for example, the Poisson distribution, where the Wiener-Charlier orthogonal expansion for the Poisson-Wiener functionals can be used [31] instead of the Wiener-Hermite orthogonal expansion used in this paper. Moreover, based on the present theory, the influence of the rough boundary on the propagation characteristics of pulse-modulated signals in a waveguide can also be investigated from the intensity transfer equation involving the time dependent term. Research on the scattering of guided waves in an optical fiber with a slightly rough boundary will also be done and reported later. For optical fibers, the effects of the rough boundary may be more important than in the case presented in this paper, due to the presence of the radiation modes and the leaky modes. On the other hand, the problem in an optical fiber will become more complicated because the hybrid boundary conditions rather than the Dirichlet condition have to be used.

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